PHASE-PLANE METHOD: A PRACTICAL APPROACH

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ABSTRACT

In this article phase-plane trajectories of stochastic processes are investigated. We discuss the problems arising when phase-plane trajectories are being plotted Then we systematically examine trajectories from different probability distributions in search for a generalized phase-plane analysis approach. Practical application of phase-plane method is given in the end of this article.

I. INTRODUCTION

Stochastic models of dynamic systems and informational processes are based on the theory of random processes [1]. Such models are used to describe real world processes, i.e. height of the wind-induced waves, combustion-chamber pressure in an engine, Brownian motion of particles, stock market fluctuations, miscellaneous artifacts and noises arising in data acquisition and transmission devices, physiological data such as temperature, pulse, respiration, blood pressure and many others.

Phase space is sometimes referred to as one of the most powerful inventions of modern science [2]. The reason of such a loud statement is that a simple two-dimensional phase space which is often called a phase plane provides a way of displaying data in a form that emphasizes the dynamic activity in a system [3].

In mathematics and physics the concept of phase space is used to describe multiple states of a system. A single phase space point describes a state of an arbitrarily complex system such as, for example, a vehicle, combustion engine, nuclear reactor or a human body. The movement of this point is called a phase trajectory. It describes evolution of the entire system in time. If we accept that system behavior is characterized by state $\xi(t)$ of the system and its rate of change is $\xi'(t) = d\xi(t)/dt$ then the plane $\left(\xi(t), \xi'(t)\right)$ will be a phase plane (subcase of phase space with only two dimensions).

II. RANDOM PROCESS

In probability theory a random function is a collection of random variables $\{\xi(s), \xi \in X, s \in S\}$ indexed by parameter s, where s belongs to an arbitrary set S, and X is a subset of states of a random variable $\xi(s)$ [4], [5].

There are many different classification criteria for random functions. The most common type of classification is based on dimensionality and continuity (or discontinuity) of space X and set S. Such classification gives two base classes of random functions:

1. Random series, often referred to as time series (set S is discrete).
2. Stochastic processes (set S is continuous).

Parameter s most often is interpreted as time. Without loss of generality we may define stochastic process as $\{\xi(t), t \in T\}$, where set T represents time. Parameter set of parameter t is considered to be $T = \{t, 0 \le t \le \infty\}$ or $T = \{t, a \le t \le b\}$, in other words it is assumed that T matches either with positive semiaxis $t \in [0; \infty)$ or with a finite interval $t \in [a; b]$ on the time axis.

III. PHASE-PLANE TRAJECTORIES OF A RANDOM PROCESS

The task of study of peculiarities of the random process' $\xi(t)$ phase trajectory on phase plane (ξ, ξ') often arises when complex dynamical systems are examined. In this approach a random process is often visualized as a phase trajectory $L(\xi, \xi', t)$ on a phase plane. Usually the phase plane trajectory of a random process carries information about both the random process $\xi(t)$ and it's dynamics in the form of its first derivative $\xi'(t)$ (derivative of process versus process plot). Such visualization is sometimes called a first-order phase-plane.

Second-order phase-plane is a plot of second versus the first derivative of a random process [6].

More rarely a phase-plot of second derivative versus random process is used [7].

If multiple signals are observed from the system, phase-plane trajectories can be drawn by plotting the signals against each other. In fact any collection of variables that are linearly independent and fully describe the system can be used to define the dimensions (or axes) of the phase space. The concept of phase *space* means that the phase trajectory can be represented in a 3-D plot with three different axes (e.g. the second derivative, the first derivative and the random process).

Although all phase portraits of each sample distribution function of a random process are unique, the phase portraits of random processes with the same distribution function are visually similar. Level crossing theory allows for the best way to describe these similarities.

IV. TIME-SERIES PLOTS VERSUS PHASE-PLANE PLOTS

On fig. 1 the relationship between a time-series plot and corresponding phase-plane trajectory is shown.

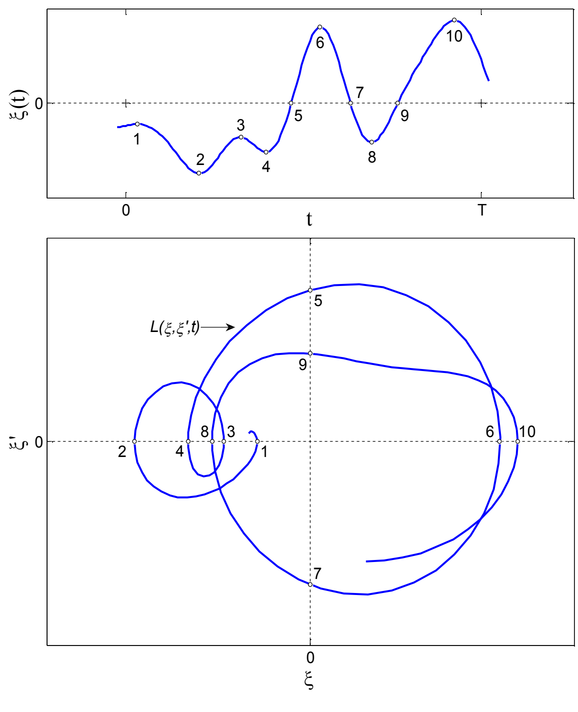

Fig. 1. Random process $\xi(t)$ (top) and its phase portrait (bottom). $L(\xi,\xi',t)$ is the phase trajectory of $\xi(t)$

Points 5 and 9 (fig. 1) reflect the process $\xi(t)$ upcrossing zero level while the phase-plane trajectory crosses the $0\xi'$ axis from left to right. Point 7 corresponds to $\xi(t)$ downcrossing zero level with the phase-plane trajectory crossing the $-0\xi'$ axis from right to left. Points 1 – 4, 6, 8 and 10 are all local extrema of process $\xi(t)$ which means that the phase-plane trajectory crosses the X-axes at those points with point 2 (minimum value) and point 10 (maximum value) being the leftmost and the rightmost trajectory points respectively. The relationship between critical points of process $\xi(t)$ and X- and Y-axes crossings of phase-plane trajectory makes it easier to retrieve the additional information stored in the phase plane and to describe this trajectory verbally.

To minimize the error of estimating the axes crossing points of the phase-plane trajectory listed above the trajectory needs to be a smooth curve. This means that the initial time-series must be converted to a smooth curve before the phase portrait is plotted. One of the methods widely used for this purpose is B-spline fitting [6], [8]. Another approach is to increase the sampling frequency either by altering the sampling frequency of the device responsible for data acquisition (i.e. an ADC) or by resampling the original time-series. Altering the data acquisition device's sampling frequency in many cases may be impossible, so upsampling filters are used instead. An example of resampling smoothing digital filter is discussed in [9].

V. PHASE-PLANE TRAJECTORIES OF COMMON PROBABILITY DISTRIBUTIONS

Phase-plane trajectory of Gaussian random process is shown on fig. 2. This phase portrait is characterized by approximately equal density of trajectory lines in all four quadrants of the plane. This fact is easily explained as Gaussian density distribution function is symmetric about the mean value.

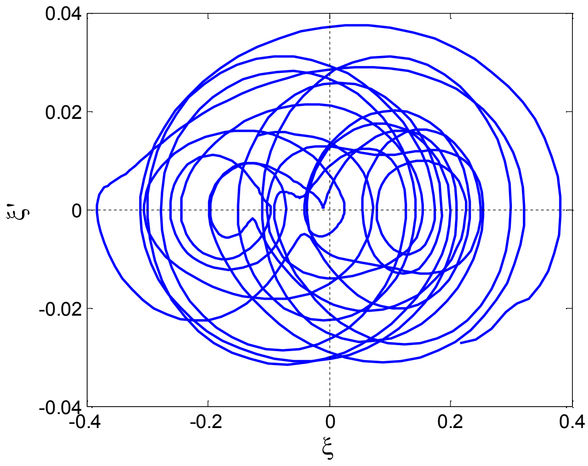

Fig. 2. Phase portrait (phase trajectory) $L(\xi,\xi',t)$ of Gaussian random process $\xi(t)$

Phase portrait of a Rayleigh random process is shown on fig. 3. The density of trajectory lines is slowly descending with increase of X-axis values according to the long right tail of Raleigh density distribution function.

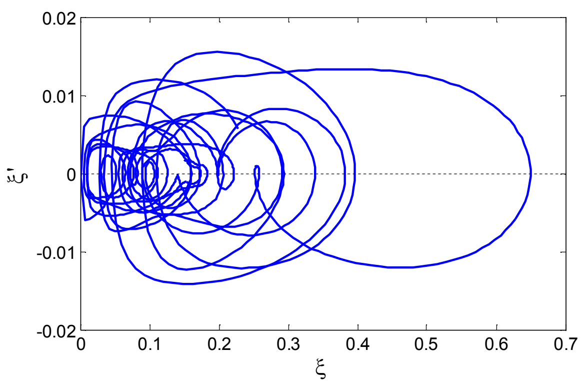

Fig. 3. Phase trajectory of a Rayleigh random process $\xi(t)$

In contrast to phase-plane trajectory of a Rayleigh random process the trajectory shown on figure 4 has the highest density of lines immediately around zero point with the density rapidly descending towards positive infinity. This trajectory belongs to a random process with exponential PDF.

Taking into account the difference between Rayleigh and Exponential probability density functions we conclude that in general phase-plane trajectory lines of a random process are concentrated in the vicinity of the mode of the stochastic process.

The main difference between the Rayleigh distribution discussed above and the Maxwell distribution is that the latter has a longer left and shorter right density function tails making it closer to Gaussian distribution. This distinguishing characteristic is clearly visible on the phase portrait depicted on fig. 5.

Finally, as shown on fig. 6 the log-normal distribution is characterized by extremely long probability density function tale.

Mathematical description of phase-plane trajectories of random processes based on level-crossing theory makes it possible to quantify the differences between probability distributions discussed above [5],[10], [11].

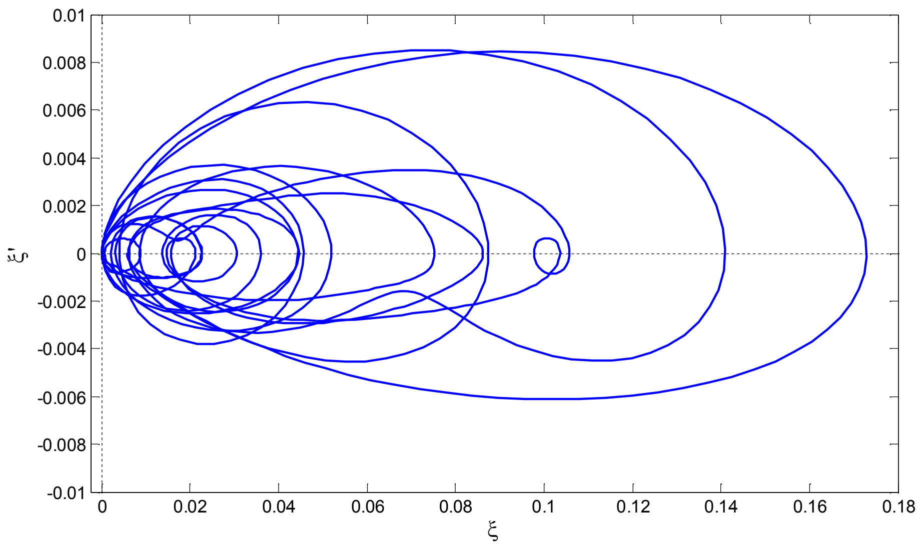

Fig. 4. Phase trajectory of a random process $\xi(t)$ with exponential distribution

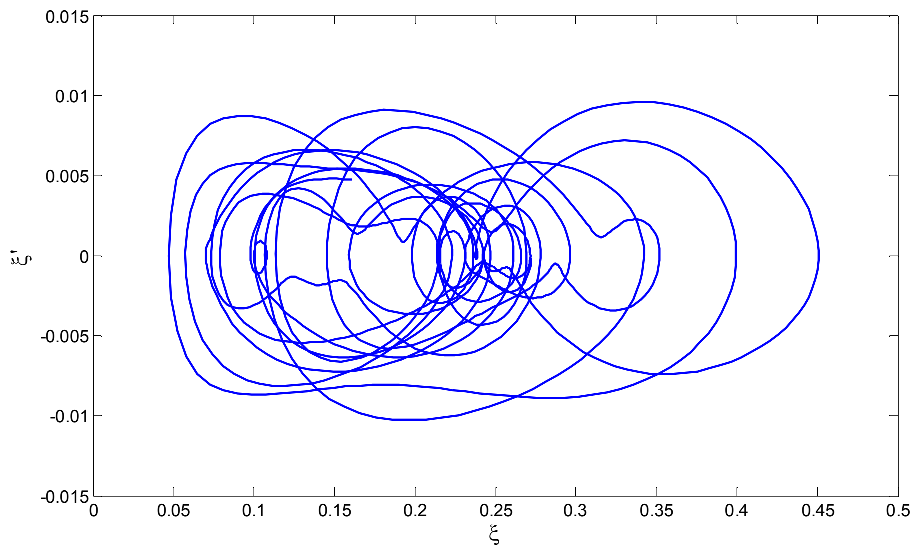

Fig. 5. Phase trajectory of a Maxwell random process $\xi(t)$

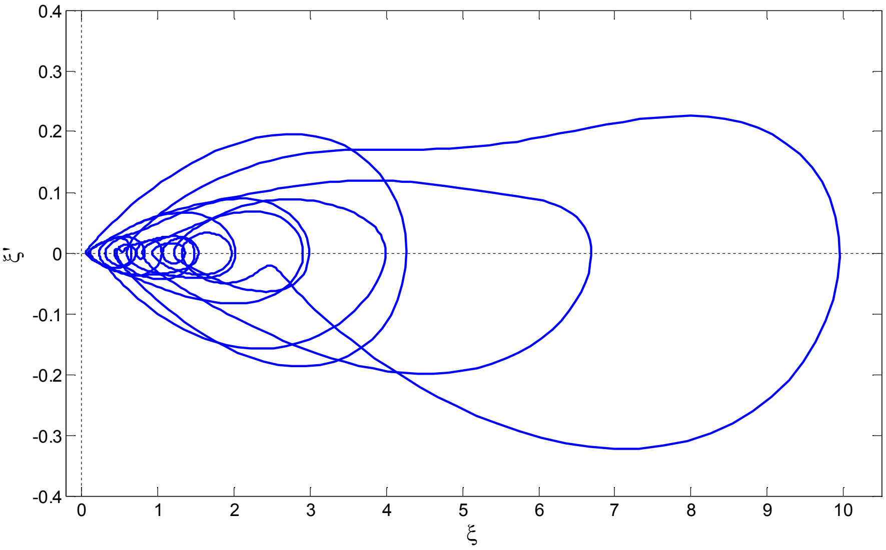

Fig. 6. Phase trajectory of a random process $\xi(t)$ with log-normal distribution

VI. APPLICATION IN THE FIELD OF MEDICINE: ECG ANALYSIS

A fragment approximately 5 seconds length of an ECG record of young 23 years old healthy male and a phase-plane trajectory of this ECG signal are both shown on figure 7. The fragment is taken from record *f2y02* stored in "*Fantasia Database*" of PhysioNet research resource for complex physiologic signals [12].

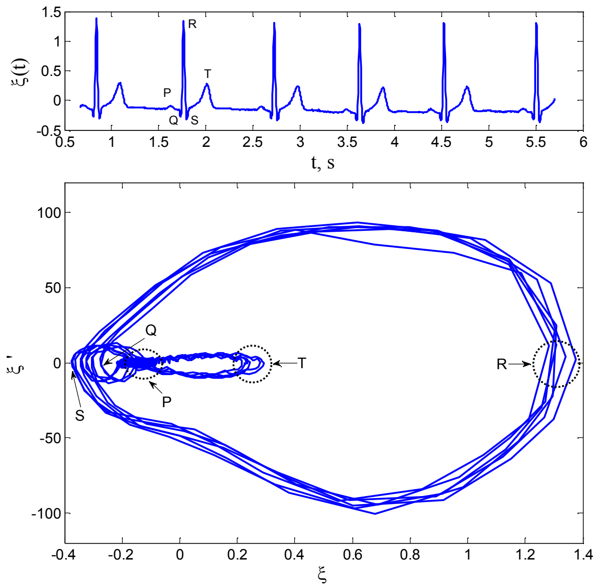

Fig. 7. Fragment of ECG signal of a healthy human (top) and its phase trajectory (bottom)

ECG analysis is mainly based on the following variables: minimum and maximum values of the signal (so called QRS-complex) and time intervals between them. By applying previously described method [10] it is possible to define several tolerance boxes, i.e. areas $\Omega_1, \Omega_2, \ldots, \Omega_n$ on a phase plane, for each variable in study.

The group of largest oval-shaped figures with greatest diameter on the phase portrait (fig. 7 bottom) corresponds to the QRS-complex of ECG signal. Group of small horizontally stretched ovals with indistinct borders in the left part of phase portrait is produced by ST segment of ECG. The fuzzy area to the left of ST segment with lots of phase trajectory intersections actually looking more like a spot corresponds to the PR-segment of the electrocardiogram.

So phase-plane ECG representation with the help of level-crossing theory allows to detect changes in cardiac performance.

VII. OTHER AREAS OF APPLICATION

Study of phase trajectories of random processes opens a prospect of effective analysis and control of complex dynamical signals [5], which are hard to formalize using other methodological approaches. Such complex systems and signals are found in medicine [3], [8], biology [13], technics [14], [15] and other areas [6], [16].

A good example can be found in the study of a work of USA researches [3] concerning investigation of EEG phase-plane trajectories associated with the transition from the interictal state to clinical seizures. It was shown that the use of traditional EEG waveforms (time-series plots) together with phase-plane analysis for epileptic seizure onset prediction may increase the efficiency and accuracy of such predictions. Unfortunately to apply this method in medical practice it is necessary to train doctors, i.e. electroencephalographers (EEGers), to read and decode a completely new form of EEG representation, a trajectory of biophysical process on a phase-plane. As a result the conclusion of this research was based on subjective measures.

Level-crossing theory can give a formal description of phase trajectories of a random process such as EEG signal [10]. This means that the task of manual analysis of phase-plane trajectories by EEGer can be automated. The EEG specialist will have to deal only with valuable numeric parameters of phase-plane trajectory of EEG signal or even with a decision suggestion made by automated analysis machine. If a mathematical model of a process is present the algorithms for calculating level-crossing characteristics are easily implemented both on software and hardware levels.

Phase-plane signal representation together with level-crossing theory may be also used in telemetry data analysis. The telemetry data may come from a space vehicle, an aircraft, a nuclear reactor or a medical device (biotelemetry, see the above discussion of ECG and EEG data).

In [10] an application of phase-plane analysis in reliability theory is shown. A similar approach may be used for analysis of risk situations in economics, business, engineering and other risk-prone areas.

With phase-plane method it is easy to find segments with different characteristics of a process and the level-crossing theory makes it possible to find transition points where the process switches from one state to another. Thus, the described method may be used for analysis, prediction, diagnostic, decision making tasks.

VIII. CONCLUSION

Phase-plane is a good method for examination of the detailed structure of stochastic processes. From figures 2-6 it is clear that visual appearance of a phase-plane trajectory of a random process greatly depends on the probability distribution of this process.

In paragraph IV a relationship between phase-plane trajectories and level-crossing characteristics of a random process was shown. This relationship makes it possible to describe phase-plane trajectories by means of mathematical language using the level-crossing theory.

Phase-plane method seems to be promising for solving the task of automating data analysis in such areas as medicine (analysis of medical data like ECG, EEG, etc.), telemetry data analysis and risk estimation.

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