MAIN MULTIDIMENSIONAL NON-GAUSSIAN DENSITIES OF DISTRIBUTIONS USED IN MODELING OF TECHNICAL SYSTEMS

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I. INTRODUCTION

Traditional description of any random processes, including non-Gaussian, is the description as a multidimensional densities of probability distribution [1 – 6]. But finding a multivariate density of distribution for many real processes is almost impossible. Moreover, checking the adequacy of a hypothetical multi-dimensional density of distribution and density of distribution of real signals, is essentially impossible, because the definition of multivariate density required too large sampling of signals obtained for the same experimental conditions. Therefore, in practice, limited to the requirements of the adequacy of the marginal distribution laws to the distribution law of real signals and the adequacy of the correlation and spectral characteristics of models and signals.

With imposed restrictions, experimental verification of the adequacy of accepted mathematical models to real signals is possible, because during the experiment it is possible to obtain sufficiently reliable estimates of one-dimensional density of distribution and of the correlation function or spectrum of the observed signals. These difficulties of obtaining multidimensional non-Gaussian densities of distribution and the sufficiency of two-dimensional densities for the experimental verification of the adopted models (verification model) make it necessary to be limited in the theoretical researches with two-dimensional densities of distribution [5, 7]. Therefore, in this work only two-dimensional distributions are given which have been used for modeling of non-Gaussian processes and fields [3, 5, 7, 8].

II. TWO-DIMENSIONAL NON-GAUSSIAN DENSITIES OF DISTRIBUTION

1. Nakagami distribution (m-distribution).

This distribution is used to describe the joint density of distribution of envelope of radio signals

$$w(U_1,U_2,m,\sigma_1,\sigma_2,r_2)=(1-r_2)^m\sum_{n=0}^{\infty}\frac{(m)_n}{n!}r_2^n w(U_1,m,\sigma_1)\times$$
$$\times w(U_2,m,\sigma_2), \tag{1}$$

where

$$w(U_i,m\ ,\sigma_i)=\left\{\frac{2\sigma_i^{-(m+n)}\left(m+n^{m+n}\right)}{\Gamma(m+n)(1-r_2)^{m+n}}\right\}\times$$
$$U_i^{2(m+n)-1}\exp\left(-\frac{(m+n)U_i^2}{\sigma_i(1-r_2)}\right),\ \mathrm{i}=1,2; \tag{2}$$

$$(\mathrm{m})_n=m(m+1),\ldots,(m+n-1);\quad (\mathrm{m})_0=1;$$

where r_2 – is the coefficient of mutual correlation between the squares of the envelopes U_1 and U_2, m and σ_i – parameters.

If envelopes U_1 and U_2 are not correlated and $m_1 \neq m_2$, then

$$w(U_1,U_2,m_1,m_2,\sigma_1,\sigma_2)=\frac{4}{\Gamma(m_1)\Gamma(m_2)}\left(\frac{m_1}{\sigma_1}\right)^{m_1}\times$$
$$\times\left(\frac{m_2}{\sigma_2}\right)^{m_2}U_1^{2m_1-1}U_2^{2m_2-1}\exp\left\{-\frac{m_1}{\sigma_1}U_1^2-\frac{m_2}{\sigma_2}U_2^2\right\}. \tag{3}$$

2. Two-dimensional distribution of the Rayleigh-Rice or generalized Rayleigh distribution.

This distribution describes a two-dimensional density of distributions of the sum of a stationary Gaussian process $n(t)\sim N(0,\sigma_*^2)$ and deterministic signal $s(t)=a(t)\cos(\omega_0 t)$. Since in this case, the total process will be nonstationary, the distribution of its multidimensional values will also be nonstationary:

$$w(\xi_1,\xi_2,\tau,t)=\frac{\xi_1\xi_2}{\sigma^4(1-R^2(\tau))}\times$$
$$\times\exp\left\{-\frac{\xi_1^2+\xi_2^2+a_1^2+a_2^2-2a_1a_2R(\tau)}{2\sigma^2(1-R^2(\tau))}\right\}\times$$
$$\times\sum_{m=0}^{\infty}\varepsilon_m I_m\left[\frac{R(\tau)\xi_1\xi_2}{\sigma^2(1-R^2(\tau))}\right]I_m\left[\frac{a_1-R(\tau)\xi_1\xi_2a_2}{\sigma^2(1-R^2(\tau))}\right]\times I_m\left[\frac{a_2-R(\tau)\xi_1\xi_2a_1}{\sigma^2(1-R^2(\tau))}r_2\right], \quad (4)$$

where $\xi_1>0,\xi_2>0,\varepsilon_0=1,\varepsilon_m=2,m>0,R(\tau)$ – is normalized correlation function.

If the deterministic signal is a harmonic oscillation of frequency ω_0 and amplitude U_0, then $a_1=a_2=U_0$ and the density of probability distribution of a stationary random process $\xi(t)$ becomes

$$w(\xi_1,\xi_2,\tau)=\frac{\xi_1\xi_2}{\sigma^4(1-R^2(\tau))}\exp\left\{-\frac{\xi_1^2+\xi_2^2}{2\sigma^2(1-R^2(\tau))}\right\}\times$$
$$\times\exp\left\{-\frac{U_0^2}{\sigma^2(1+R(\tau))}\right\}\sum_{m=0}^{\infty}\varepsilon_m I_m\left[\frac{R(\tau)\xi_1\xi_2}{\sigma^2(1-R^2(\tau))}\right]\times$$
$$\times I_m\left[\frac{U_0\xi_1}{\sigma^2(1+R(\tau))}\right]I_m\left[\frac{U_0\xi_2}{\sigma^2(1+R(\tau))}\right],\xi_1>0,\xi_2>0.$$

When $\tau\to\infty, R(\tau)\to 0$ expression (5) simplifies

$$w(\xi_1,\xi_2)=\frac{\xi_1}{\sigma^2}\exp\left\{-\frac{\xi_1^2+a_1^2}{2\sigma^2}\right\}I_0\left(\frac{\xi_1a_1}{\sigma^2}\right)I_0\left(\frac{\xi_2a_2}{\sigma^2}\right)\frac{\xi_2}{\sigma^2}\times\exp\left\{-\frac{\xi_2^2+a_2^2}{2\sigma^2}\right\}. \quad (5)$$

3. Density of probability distribution of Laplace.

$$w(\xi_1,\xi_2,\tau_2)=w(\xi_1,|\,\xi_2,\tau_2)w(\xi_2), \quad (6)$$

$w(\xi_1\,|\,\xi_2,\tau)$ – conditional density of probability distribution; $w(\xi_2)$ - dimensional density of probability distribution;

$$w(\xi_1\,|\,\xi_2,\tau)=\frac{\sqrt{2}}{\sqrt{\sigma_\xi^2(1-R^2(\tau))}}\exp\left\{\frac{\sqrt{2}|\xi_1-R(\tau)\xi_2|}{\sqrt{\sigma_\xi^2(1-R^2(\tau))}}\right\};$$
$$w(\xi_2)=\frac{\sqrt{2}}{\sigma^2}\exp\left\{-\frac{\sqrt{2}|\xi_1|}{\sigma^2}\right\}. \quad (7)$$

4. Log-normal distribution.

This distribution is most often used to describe the joint density of probability distribution of envelopes of input signals (information and interfering) of onboard radioelectronic systems:

$$w(U_1,U_2,\sigma_1^2,\sigma_2^2,\mu_1,\mu_2,r_{12})=\frac{U_1^{-1}U_2^{-1}}{2\pi\sigma_1\sigma_2\sqrt{1-r_{12}^2}}\times$$
$$\exp\left\{-\frac{1}{2(1-r_{12}^2)}\left[\frac{(\ln U_1-\mu_1)^2}{\sigma_1^2}+\frac{(\ln U_2-\mu_2)^2}{\sigma_2^2}-\frac{2r_{12}}{\sigma_1\sigma_2}(\ln U_1-\mu_1)(\ln U_2-\mu_2)\right]\right\}, \quad (8)$$

where σ_1^2, σ_2^2, μ_1, μ_2 – are dispersions and mean values of logarithms of envelopes components,

$$\sigma_i^2=\left\langle(\ln U_i-\mu_i)^2\right\rangle;\mu_i=\langle\ln U_i\rangle; \quad (9)$$

r_{12} – cross-correlation coefficient between the logarithms of envelopes components.

Average value of envelopes

$$m_{U_i}(\mu_i,\sigma_i)=\exp\left(\frac{\sigma_i^2}{2}+\mu_i\right),i=1,2. \quad (10)$$

Dispersion of envelopes

$$\sigma_{U_i}^2(\mu_i,\sigma_i)=\exp(2\mu_i+\sigma_i^2)\left[\exp(\sigma_i^2)-1\right], \quad (11)$$

where

$$\begin{cases}\mu_i(m_{U_i},\sigma_{U_i}^2)=\ln\left(\dfrac{m_{U_i}}{\sqrt{1+\sigma_{U_i}^2m_{U_i}^{-2}}}\right);\\ \sigma_i^2(m_{U_i},\sigma_{U_i}^2)=\ln(1+\sigma_{U_i}^2m_{U_i}^{-2}).\end{cases} \quad (12)$$

The function of mutual correlation between the envelopes in the matching points in time is

$$B_{U_1,U_2}^{(\tau)}=\int_0^\infty\int_0^\infty(U_1-m_{U_1})(U_2-m_{U_2})w(U_1U_2)dU_1dU_2=$$
$$=\exp\left(\mu_1+\frac{\sigma_1^2}{2}\right)\exp\left(\mu_2+\frac{\sigma_2^2}{2}\right)\{\exp(r_{12}\sigma_1\sigma_2)-1\}.$$

Correlation coefficient

$$r_{12}(m_{U_1},\sigma_{U_1}^2,r_{U_1U_2})=\frac{B_{U_1U_2}}{\sigma_{U_1}\sigma_{U_2}}=\frac{\ln\left\{1+\dfrac{r_{U_1U_2}\sigma_{U_1}\sigma_{U_2}}{m_{U_1}m_{U_2}}\right\}}{\sqrt{\ln\left(1+\dfrac{\sigma_{U_1}^2}{m_{U_1}^2}\right)\ln\left(1+\dfrac{\sigma_{U_1}^2}{m_{U_1}^2}\right)}}.$$

One-dimensional log-normal probability distribution has the form

$$w(U,\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2U}}\exp\left\{-\frac{(\ln U-\mu)^2}{2\sigma^2}\right\}. \quad (13)$$

Asymmetry coefficient

$$k_a=\sqrt{\exp(\sigma^2)-1}(\exp(\sigma^2)+2). \quad (14)$$

Coefficient of kurtosis

$$k_э=3+\left(\exp(\sigma^2)-1\right)\left(\exp(3\sigma^2)+3\exp(2\sigma^2)+6\exp(\sigma^2)+6\right), \quad (15)$$

5. Two-dimensional density of probability distribution of envelope of stationary narrow-band normal random process

$$w(\xi_1,\xi_2,\tau)=\frac{\xi_1\xi_2}{\sigma^4(1-R^2(\tau))}\exp\left\{-\frac{\xi_1^2+\xi_2^2}{2\sigma^2(1-R^2(\tau))}\right\}\times$$
$$\times I_0\left\{\frac{R(\tau)\xi_1\xi_2}{\sigma^2(1-R^2(\tau))}\right\}. \quad (16)$$

where

$$\tau\to 0;\ R(\tau)\to 0;\ w(\xi_1,\xi_2,\infty)=$$
$$=\frac{\xi_1}{\sigma^2}\exp\left(-\frac{\xi_1^2}{2\sigma^2}\right)\frac{\xi_2}{\sigma^2}\exp\left(-\frac{\xi_2^2}{2\sigma^2}\right)$$

(17)

Density of probability distribution of transition is widespread in applied problems.

$$w(\xi_2,\tau\mid\xi_1)=\frac{w(\xi_1,\xi_2,\tau)}{w(\xi_1)}=\frac{\xi_2}{\sigma^2(1-R^2(\tau))}$$
$$\exp\left\{-\frac{R^2(\tau)(\xi_1^2+\xi_2^2)}{2\sigma^2(1-R^2(\tau))}\right\}I_0\left\{\frac{R(\tau)\xi_1\xi_2}{\sigma^2(1-R^2(\tau))}\right\}, \quad (18)$$
$$\xi_1>0,\xi_2>0;R(\tau)=\exp(-\beta|\tau|).$$

Correlation function of envelope of narrow-band stationary Gaussian random process

$$B_U(\tau)=\frac{\pi}{2}\sigma^2\left\{1+\frac{1}{4}R^2(\tau)+\sum_{n=2}^{\infty}\frac{[(2n-3)!!]^2}{2^{2n}(n!)^2}R^{2n}(\tau)\right\}=$$
$$=\sigma^2\{2E[R(\tau)]+[1-R^2(\tau)]K[R(\tau)]\}, \quad (19)$$

where K and E – are complete elliptic integrals of first and second kind, respectively.

III. CONCLUSION

Here are given the simplest two-dimensional densities of probability distribution which are used for synthesis and analysis of technical systems. From this review, becomes apparent complexity of the synthesis of scalar random processes with such distributions of multidimensional values or parameters. Even more difficult is the problem of synthesis of the vector (multidimensional) random processes and fields.Therefore, when choosing a model, simplicity of this model is important, especially when the simplicity of the model allows one to synthesize the most efficient, in the sense of optimal, algorithms of modeling.

Requirements of adequacy of the non-Gaussian two-dimensional models to real signals, and its simplicity are mutually contradictory, a compromise solution to this problem depends strongly on the specific subject area. In this paper subject area is limited to input signals of on-board data-measuring systems and control systems. Limitation of interfering signals caused by reflections of the probing signal from the underlying surface of the earth and sea, from hydrometeors and signals of interfering electromagnetic field produced by a similar radio-electronic means, allows to narrow the the task of selection of mathematical model and to stop the choice for a multidimensional log-normal density of distribution. This model allows one to synthesize relatively simple non-linear forming filter, which output signal has a given correlation function, and given (log-normal) probability distribution.

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