SIMULATION OF GAUSSIAN AND NON-GAUSSIAN ANISOTROPIC RANDOM FIELDS

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Abstract

The paper discusses simulation algorithms for 1D, 2D, and 3D anisotropic random fields. The algorithms are the natural extension of the synthesis method used to simulate random vectors and matrices with given statistic properties.

The synthesis methods [1] discuss for Nonlinear Multichannel Discreet Forming Filters (NMDFF) used to simulate random vectors and matrices with given statistic properties can be naturally extended to simulate Gausian and non-Gaussian random fields with a space-domain co-ordinate anisotropy. The synthesis method presented in this paper has the only one limitation for direct application: the limitation to factorization of correlation functions on their space-domain co-ordinates for the non-Gaussian field generating the respective Gaussian field (generally, the anisotropic one). Nevertheless, this limitation is relatively of small importance in practice, as the results of experimental studies of the fields are quite often represented by their cross-sections of their space-domain correlation function, one for each co-ordinate and time. That is why the introduced limitation does not prevent a user from building a statistic field model not contradicting to the experimental data.

We need to emphasize that the requirement to the space-time domains correlation function being factorable is necessary only to the generating Gaussian field. The non-Gaussian field being simulated can have a non-factorable space-time domains correlation function. Moreover, the generating Gaussian field in practical applications can be expanded into its orthogonal components, which are statically independent due to the field being normal. In this case, the factorization is natural and, therefore, does not limit the problem of simulating the generating field.

In [1] assumes that, for NMDFF synthesis, the normalized time-domain statistic properties for the processes in channel forming filters are the same. This assumption drastically increases the speed of simulation algorithms, whilst it is not too much restrictive for practical applications of the synthesized algorithms; as such assumption is true for many practical cases. Frequently, this assumption is simply assumed and even not mentioned. That is why we assume that this condition is always true. We should mention nevertheless that the process average values and variances can vary, which is used in practical cases. We make the same assumption for space-domain co-ordinate (that the normalized statistical properties are the same along all field co-ordinates).

We confine ourselves to the explicit notation of simulation algorithms for 1D (vector), 2D (matrix), and 2D (space-domain) fields, as these field types are used for mathematical models of real physical processes. For example, such fields describe, respectively, active jam (and informational) signals, underlying terrain interferences (from land and see), dispersive interferences (meteorics and dipole dipole scatterer fields).

For the algorithms simulating non-Gaussian fields, we introduce the following designations: V are the elements of the non-Gaussian fields, U the elements of the generating Gaussian field, (X, Y, Z) orthogonal field co-ordinates of Field U, $r^{(X)}, r^{(Y)}, r^{(Z)}$ the respective normalized correlation functions of the generating Gaussian field (elements of correlation matrices), $f(.)$ a functional transform corresponding to the required distribution density of the field being simulated, ξ normal mutually independent random (pseudo-random) values with zero average and unit variances. All the introduced values can generally have an arbitrary number of indices. In our particular case, we confine ourselves to not more than time-variable 3D fields; the number of indices is not more than 4, one always being time [2].

The simulation algorithm for a 1D (vector) field in these designations can be represented as:

$$\begin{cases} X_{i,t} = -\sum_{l=1}^{i-1} (D_{l,i}^{(X)} / D_{l,l}^{(X)}) \cdot X_{l,t} + \sqrt{D_i^{(X)} / D_{i-1}^{(X)}} \cdot \xi_{i,t} , \\ U_{i,t} = \sum_{s=1}^{N} a_s \cdot U_{i,t-s} + \sum_{d=0}^{N-1} b_d \cdot X_{i,t-d} , \\ \widetilde{V}_{i,t} = f(\overline{U}_{i,t} + \sigma_{i,t} \cdot U_{i,t}) = \overline{U}'_{i,t} \cdot \exp(\sigma_{i,t} \cdot U_{i,t}) , \\ \qquad\qquad i = 1,2,...M^{(X)}, \end{cases} \quad (1)$$

where $X_{1,t} = \xi_{1,t}$, $\forall t$, $t = \ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots$, $D_1^{(X)} = D_0^{(X)} = 1$, $D_{l,l}^{(X)} = D_{l-1}^{(X)}$, $l = 1, 2, \ldots M^{(X)}$.

$$D_l^{(X)} = \begin{vmatrix} 1 & r_1^{(X)} & r_2^{(X)} & \cdots & r_{l-1}^{(X)} \\ r_1^{(X)} & 1 & r_1^{(X)} & \cdots & r_{l-2}^{(X)} \\ r_2^{(X)} & r_1^{(X)} & 1 & \cdots & r_{l-3}^{(X)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{l-1}^{(X)} & r_{l-2}^{(X)} & r_{l-3}^{(X)} & \cdots & 1 \end{vmatrix}$$

The distribution density $\widetilde{V}_{i,t}$ is determined by the functional transform $f(.)$.

We can interpret Algorithm (2) as follows:

1. At each fixed *t*, the topmost expression in (1) transforms the vector of the mutually independent normally distributed random values $\boldsymbol{\xi}_{M^{(X)},t} = (\xi_{1,t}, \xi_{2,t}, \ldots \xi_{M^{(X)},t})$ with zero average and unit variances into the vector $\mathbf{X}_{M^{(X)},t} = (X_{1,t}, X_{2,t}, \ldots X_{M^{(X)},t})$ of normally distributed random values with zero average and unit variances, too, but with the normalized correlation matrix $\mathbf{D}_{M^{(X)}}^{(X)} = \left\| D_{M^{(X)}}^{(X)} \right\|$, the matrix $\mathbf{D}_{M^{(X)}}^{(X)}$ being independent on *t*. We write such transform as:

$$\boldsymbol{\xi}_{M^{(X)},t} \overset{\mathbf{D}_{M^{(X)}}^{(X)}}{\Rightarrow} \mathbf{X}_{M^{(X)},t}. \tag{3}$$

2. The middle expression in (1) transforms each of the elements of the vector $\mathbf{X}_{M^{(X)},t} = (X_{1,t}, X_{2,t}, \ldots X_{M^{(X)},t})$ into the sequence $\mathbf{U}_{i,t} = (\ldots -2, -1, 0, 1, 2, 3, \ldots)$ by a linear discreet forming filter [1] with the coefficients $(\mathbf{a}_N, \mathbf{b}_N) = (a_1, a_2, \ldots a_N; b_0, b_1, \ldots b_{N-1})$ which are the normal time-domain correlation function of the sequence $r^{(B)}(t_1, t_2) = r^{(B)}(|t_1 - t_2|) = r^{(B)}(\tau)$ for each element of the vector $\mathbf{X}_{M^{(X)},t}$, being independent from the first index of the vector element $\mathbf{X}_{M^{(X)},t}$ due to the assumptions made. Therefore, this transform forms a vector normal process, or, in other words, forms $M^{(X)}$ normal correlated processes, which individual time-domain correlation function being equal to $r^{(B)}(\tau)$, and the cross-correlation function of the processes $X_{i,t_1} = X_i(t_1)$ and $X_{j,t_2} = X_j(t_2)$ equal to $r^{(B)}(|t_1 - t_2|) \cdot r^{(X)}(|i - j|)$, $i, j = 1, 2, \ldots M^{(X)}$. We designate this transform as:

$$X_{M^{(X)},t} \overset{(\mathbf{a}_N, \mathbf{b}_N)}{\Longrightarrow} \mathbf{U}_{M^{(X)},t} \tag{4}$$

3. The expression in the third line on (1) determines the non-linear functional transform of $M^{(X)}$ normal stationary and stationary-coupled processes into $M^{(X)}$ not-Gaussian non-stationary processes $V_{i,t_1} = V_i(t_1)$, or in other works, into a non-Gaussian vector process $\widetilde{\mathbf{V}}_{M^{(X)},t}$. The distribution density, correlation, and cross-correlation functions of the processes $\widetilde{\mathbf{V}}_{M^{(X)},t}$ can be evaluated through the expressions from [2]. We designate this non-linear transform, that includes the parameters

$$(\overline{\mathbf{U}}_{M^{(X)},t}, \boldsymbol{\sigma}_{M^{(X)},t}) = (\overline{U}_{1,t}, \overline{U}_{2,t}, \ldots \overline{U}_{M^{(X)},t}; \sigma_{1,t}, \sigma_{2,t}, \ldots \sigma_{M^{(X)},t})$$

as:

$$\mathbf{U}_{M^{(X)},t} \overset{f(.,\overline{\mathbf{U}}_{M^{(X)},i}, \boldsymbol{\sigma}_{M^{(X)},i})}{\Rightarrow} \widetilde{\mathbf{V}}_{M^{(X)},t}$$

We can re-write Algorithm (3.46) in these designations as:

$$\boldsymbol{\xi}_{M^{(X)},t} \overset{\mathbf{D}_{M^{(X)}}^{(X)}}{\Rightarrow} \mathbf{X}_{M^{(X)},t} \overset{(\mathbf{a}_N, \mathbf{b}_N)}{\Rightarrow} \mathbf{U}_{M^{(X)},t} \overset{f(.,\overline{\mathbf{U}}_{M^{(X)},i}, \boldsymbol{\sigma}_{M^{(X)},i})}{\Rightarrow} \widetilde{\mathbf{V}}_{M^{(X)},t} \tag{5}$$

We can, using (5), write down simulation algorithms for fields of higher dimensions purely formally, by adding new indices.

We can formally derive the simulation algorithm of a 2D field (a matrix) from (5) by introducing a co-ordinate Y, an additional transform $\mathbf{X}$ into $\mathbf{Y}$, and respective indexing of the designations:

$$\boldsymbol{\xi}_{M^{(X)},M^{(Y)},t} \overset{\mathbf{D}_{M^{(X)}}^{(X)}}{\Rightarrow} \mathbf{X}_{M^{(X)},M^{(Y)},t} \overset{\mathbf{D}_{M^{(Y)}}^{(Y)}}{\Rightarrow} \mathbf{Y}_{M^{(X)},M^{(Y)},t} \overset{(\mathbf{a}_N, \mathbf{b}_N)}{\Rightarrow} \mathbf{U}_{M^{(X)},M^{(Y)},t} \overset{f(.,\overline{\mathbf{U}}_{M^{(X)},M^{Y},i}, \boldsymbol{\sigma}_{M^{(X)},M^{Y},i})}{\Rightarrow} \widetilde{\mathbf{V}}_{M^{(X)},M^{(Y)},t} \tag{6}$$

Equation (6) can be expanded:

$$\begin{cases} X_{i,j,t} = -\sum_{l=1}^{i-1} (D_{l,i}^{(X)} / D_{l,l}^{(X)}) \cdot X_{l,j,t} + \sqrt{D_i^{(X)} / D_{i-1}^{(X)}} \cdot \xi_{i,j,t}, \\ \qquad\qquad i = 1,2,\ldots M^{(X)}, \\ Y_{i,j,t} = -\sum_{p=1}^{j-1} (D_{p,j}^{(Y)} / D_{p,p}^{(Y)}) \cdot Y_{i,p,t} + \sqrt{D_j^{(Y)} / D_{j-1}^{(Y)}} \cdot X_{i,j,t}, \\ \qquad\qquad j = 1,2,\ldots M^{(Y)}, \\ U_{i,j,t} = \sum_{s=1}^{N} a_s \cdot U_{i,j,t-s} + \sum_{d=0}^{N-1} b_d \cdot Y_{i,j,t-d}, \\ \widetilde{V}_{i,j,t} = f(\overline{U}_{i,j,t} + \sigma_{i,j,t} \cdot U_{i,j,t}) = \overline{U}'_{i,j,t} \cdot \exp(\sigma_{i,j,t} \cdot U_{i,j,t}), \end{cases} \tag{7}$$

where similarly, (1), $X_{1,1,t} = \xi_{1,1,t}$ for $\forall t$, $t = \ldots -3,-2,-1,0,1,2,3,4,\ldots$ and for $\forall t$ $D_1^{(X)} = D_0^{(X)} = D_1^{(Y)} = D_0^{(Y)} = 1$, $D_{l,l}^{(X)} = D_{l-1}^{(X)}$, $l = 1,2,\ldots M^{(X)}$, $D_{p,p}^{(Y)} = D_{p-1}^{(Y)}$, $p = 1,2,\ldots M^{(Y)}$.

The simulation algorithm of a 3D field (a cube, or generally, a tensor) can be derived from (6) by introducing a co-ordinate Z, an additional transform $\mathbf{Y}$ into $\mathbf{Z}$, and respective indexing of the designations:

$$\boldsymbol{\xi}_{M^{(X)},M^{(Y)},M^{Z},t} \overset{\mathbf{D}^{(X)}_{M^{(X)}}}{\Rightarrow} \mathbf{X}_{M^{(X)},M^{(Y)},M^{Z},t} \overset{\mathbf{D}^{(Y)}_{M^{(Y)}}}{\Rightarrow}$$

$$\mathbf{Y}_{M^{(X)},M^{(Y)},M^{Z},t} \overset{\mathbf{D}^{(Z)}_{M^{(Z)}}}{\Rightarrow} \mathbf{Z}_{M^{(X)},M^{(Y)},M^{Z},t} \overset{(\mathbf{a}_N,\mathbf{b}_N)}{\Rightarrow}$$

$$\overset{(\mathbf{a}_N,\mathbf{b}_N)}{\Rightarrow} \mathbf{U}_{M^{(X)},M^{(Y)},M^{Z},t} \overset{f(.,\overline{\mathbf{U}}_{M^{(X)},M^{Y},i},\boldsymbol{\sigma}_{M^{(X)},M^{Y},i})}{\Rightarrow}$$

$$\widetilde{\mathbf{V}}_{M^{(X)},M^{(Y)},M^{Z},t} \qquad (8)$$

Equation (8) can be expanded:

$$\begin{cases} X_{i,j,k,t} = -\sum_{l=1}^{i-1}(D_{l,i}^{(X)} / D_{l,l}^{(X)}) \cdot X_{l,j,k,t} + \sqrt{D_i^{(X)} / D_{i-1}^{(X)}} \cdot \xi_{i,j,k,t}\,, \\ \qquad\qquad i = 1,2,\ldots M^{(X)}, \\ Y_{i,j,k,t} = -\sum_{p=1}^{j-1}(D_{p,j}^{(Y)} / D_{p,p}^{(Y)}) \cdot Y_{i,p,k,t} + \sqrt{D_j^{(Y)} / D_{j-1}^{(Y)}} \cdot X_{i,j,k,t}\,, \\ \qquad\qquad j = 1,2,\ldots M^{(Y)}, \\ Z_{i,j,k,t} = -\sum_{q=1}^{k-1}(D_{q,k}^{(Z)} / D_{q,q}^{(Z)}) \cdot Z_{i,p,k,t} + \sqrt{D_k^{(Z)} / D_{k-1}^{(Z)}} \cdot Y_{i,j,k,t}\,, \\ \qquad\qquad k = 1,2,\ldots M^{(Z)}, \\ U_{i,j,k,t} = \sum_{s=1}^{N} a_s \cdot U_{i,j,k,t-s} + \sum_{d=0}^{N-1} b_d \cdot Z_{i,j,k,t-d}\,, \\ \widetilde{V}_{i,j,k,t} = f(\overline{U}_{i,j,k,t} + \sigma_{i,j,k,t} \cdot U_{i,j,k,t}) = \\ \qquad = \overline{U}'_{i,j,k,t} \cdot \exp(\sigma_{i,j,k,t} \cdot U_{i,j,k,t})\,, \end{cases} \qquad (9)$$

where similarly (1), $X_{1,1,1,t} = \xi_{1,1,1,t}$ for $\forall t$, $t = \ldots -3,-2,-1,0,1,2,3,4,\ldots$ and for $\forall t$ $D_1^{(X)} = D_0^{(X)} = D_1^{(Y)} = D_0^{(Y)} = D_1^{(Z)} = D_0^{(Z)} = 1$, $D_{l,l}^{(X)} = D_{l-1}^{(X)}$, $l = 1,2,\ldots M^{(X)}$, $D_{p,p}^{(Y)} = D_{p-1}^{(Y)}$, $p = 1,2,\ldots M^{(Y)}$, $D_{q,q}^{(Z)} = D_{q-1}^{(Z)}$, $q = 1,2,\ldots M^{(Z)}$.

The following generalization of the simulating algorithms for higher dimensions is obvious, due to a very convenient representation in Expression (8).

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